MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963-A

SOME FACETS OF THE *TRI*-~~THREE~~-INDEX
ASSIGNMENT POLYTOPE

by

Egon Balas
and
Matthew J. Saltzman

Carnegie-Mellon University

PITTSBURGH, PENNSYLVANIA 15213

GRADUATE SCHOOL OF INDUSTRIAL ADMINISTRATION

WILLIAM LARIMER MELLON, FOUNDER
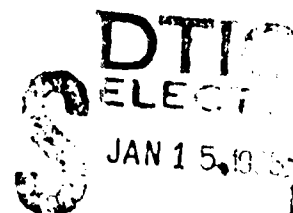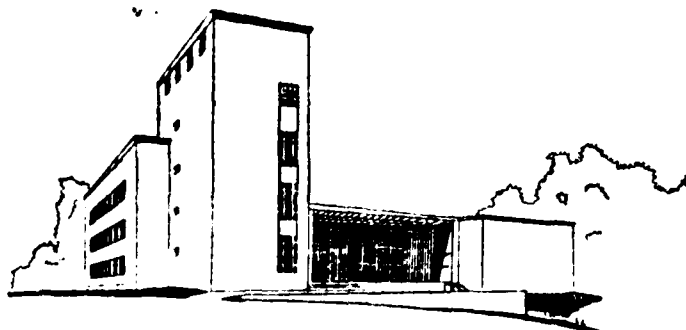
SOME FACETS OF THE *TRI*-~~THREE~~-INDEX ASSIGNMENT POLYTOPE

by

Egon Balas
and
Matthew J. Saltzman

APRIL
~~October~~ 1984

Abstract

Given three disjoint n-sets and the family of all weighted triplets that contain exactly one element of each set, the 3- index assignment (or 3-dimensional matching) problem asks for a minimum-weight subcollection of triplets that covers exactly (i.e., partitions) the union of the three sets. Unlike the common (2-index) assignment problem, the 3-index problem is NP-complete. In this paper we examine the facial structure of the 3-index assignment polytope (the convex hull of feasible solutions to the problem) with the aid of the intersection graph of the coefficient matrix of the problem's constraint set. In particular, we describe the cliques of the intersection graph as belonging to three distinct classes, and show that cliques in two of the three classes induce inequalities that define facets of our polytope. Furthermore, we give an $O(n^4)$ procedure (note that the number of variables is n^3) for finding a facet-defining clique-inequality violated by a given noninteger solution to the linear programming relaxation of the 3-index assignment problem, or showing that no such inequality exists.

1. Introduction

The *(axial) three-index assignment problem*, to be denoted AP3, also known as the (axial) *three-dimensional matching problem*, can be stated as follows: given three disjoint n-sets, I, J, and K, and a weight c_{ijk} associated with each ordered triplet $(i,j,k) \in I \times J \times K$, find a minimum-weight collection of n disjoint triplets $(i,j,k) \in I \times J \times K$.

An alternative interpretation of AP3 is as follows. A graph is *complete* if all of its nodes are pairwise adjacent. A maximal complete subgraph of a graph is a *clique*. A graph is *k-partite* if its nodes can be partitioned into k subsets such that no two nodes in the same subset are joined by an edge. It is *complete k-partite*, if every node is adjacent to all other nodes except those in its own subset. The complete k-partite graph with n_i nodes in its ith part (subset) is denoted $K_{n_1, n_2, \ldots, n_k}$.

Consider now the complete tri-partite graph $K_{n,n,n}$ with node set $S = I \cup J \cup K$, $|I| = |J| = |K| = n$. Figure 1 shows $K_{n,n,n}$ for n=2 and n=3. $K_{n,n,n}$ has 3n nodes and n^3 cliques, all of which are triangles containing exactly one node from each of the three sets I,J,K. Let (i,j,k) denote the clique induced by the node set $\{i,j,k\}$. If a weight c_{ijk} is associated with each clique (i,j,k), then AP3 is the problem of finding a minimum-weight exact clique cover of the nodes of $K_{n,n,n}$, where an exact clique cover is a set of cliques that partitions the node set S.

$K_{2,2,2}$ $K_{3,3,3}$
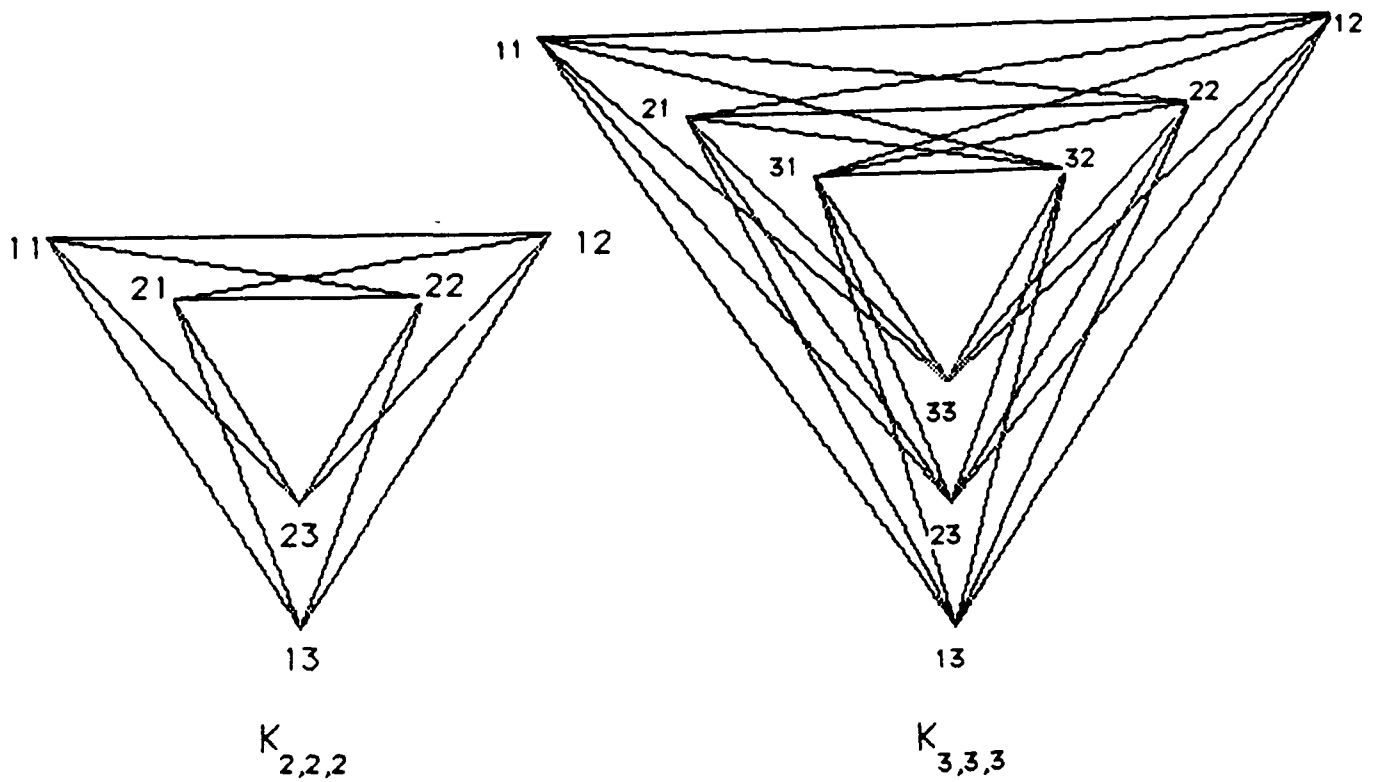
Figure 1

AP3 can be stated as a 0-1 programming problem as follows:

$$\max \quad \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} c_{ijk} x_{ijk}$$

$$\text{s.t.} \quad \sum_{j \in J} \sum_{k \in K} x_{ijk} = 1 \ \forall \ i \in I$$

$$\sum_{i \in I} \sum_{k \in K} x_{ijk} = 1 \ \forall \ j \in J$$

$$\sum_{i \in I} \sum_{j \in J} x_{ijk} = 1 \ \forall \ k \in K$$

$$x_{ijk} \in \{0,1\} \ \forall \ i,j,k$$

where I, J and K are disjoint sets with $|I| = |J| = |K| = n$. The coefficient matrix of AP3 for the case n=3 is shown in figure 2.

$$\begin{bmatrix} 111111111 & & \\ & 111111111 & \\ & & 111111111 \\ 111 & 111 & 111 \\ & 111 & 111 & 111 \\ & & 111 & 111 & 111 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$
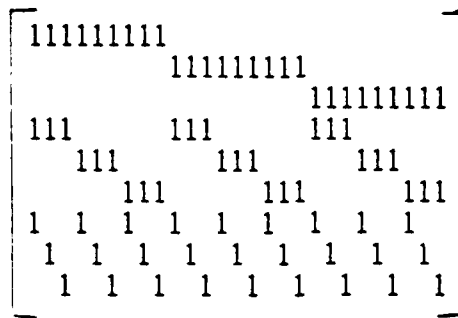
Figure 2

One interpretation of the constraints is in terms of an nXnXn cube in three-space made up of n^3 unit cubes, each to be assigned a value of zero or one so that the following conditions hold: if the cube is viewed as a set of j,k-planes stacked up in the direction of the i-axis, the first set of constraints requires that the total value of the variables in each such plane be exactly one; and a similar interpretation holds for i,k-planes (the second group of constraints, corresponding to the j-axis) as well as for i,j-planes (the third group, corresponding to the k-axis). Figure 3 illustrates this for n=3.
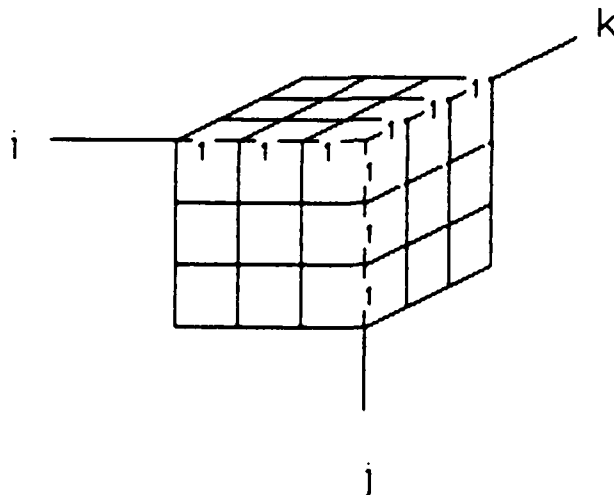
Figure 3

We will denote by $AP3_n$ the (axial) 3-index assignment problem of *order* n (i.e., defined for n-sets), by A_n the coefficient matrix of its constraint set, and by I_n, J_n, K_n the 3 associated index sets. The row and column index sets of A_n will be denoted by R_n and S_n respectively. Clearly, $|R_n| = |I_n| + |J_n| + |K_n| = 3n$ and $S_n = |I_n| \times |J_n| \times |K_n| = n^3$.

In terms of $K_{n,n,n}$, A_n is the incidence matrix of nodes versus cliques (triangles): it has a row for every node and a column for every clique of $K_{n,n,n}$.

As usual, the *support* of a (row or column) vector is understood to mean the index set of its nonzero components. Each element of S (that indexes a column of A_n and a clique of $K_{n,n,n}$) will also be used to denote the support of the given column of A_n and the node set of the given clique (triangle) of $K_{n,n,n}$. Thus, if a^S has support (i,j,k) (i.e., if clique s of $K_{n,n,n}$ has node set $\{i,j,k\}$), we will write $s = (i,j,k)$ or $a^S = a^{ijk}$, meaning that column a^S has ones in positions $i\varepsilon I$, $j\varepsilon J$ and $k\varepsilon K$.

AP3 is a special case of the (axial) *3-dimensional transportation problem*, in which the right-hand sides of the constraints can be any positive integers, the sets I,J,K are not necessarily equal in size, and the integrality constraints are relaxed. This is in turn a generalization of the well-known transportation problem, a special case of which is the simple assignment problem.

Our problem is called *axial* to distinguish it from another 3-dimensional assignment problem, called *planar*, which can be formulated as follows:

$$\max \quad \sum_{i\varepsilon I}\sum_{j\varepsilon J}\sum_{k\varepsilon K} c_{ijk}x_{ijk}$$

$$\text{s.t.} \quad \sum_{i\varepsilon I} x_{ijk} = 1 \ \forall \ j\varepsilon J, k\varepsilon K$$

$$\sum_{j\varepsilon J} x_{ijk} = 1 \ \forall \ i\varepsilon I, k\varepsilon K$$

$$\sum_{k\varepsilon K} x_{ijk} = 1 \ \forall \ i\varepsilon I, j\varepsilon J$$

$$x_{ijk} \in \{0,1\} \; \forall \; i,j,k$$

The coefficient matrix of the planar problem for n=3 is shown in figure 4.

```
⎡ 111                                              ⎤
⎢    111                                           ⎥
⎢       111                                        ⎥
⎢          111                                     ⎥
⎢             111                                  ⎥
⎢                111                               ⎥
⎢                   111                            ⎥
⎢                      111                         ⎥
⎢                         111                      ⎥
⎢ 1  1  1                                          ⎥
⎢   1  1  1                                        ⎥
⎢     1  1  1                                      ⎥
⎢           1  1  1                                ⎥
⎢             1  1  1                              ⎥
⎢               1  1  1                            ⎥
⎢                     1  1  1                      ⎥
⎢                       1  1  1                    ⎥
⎢                         1  1  1                  ⎥
⎢ 1           1           1                        ⎥
⎢   1           1           1                      ⎥
⎢     1           1           1                    ⎥
⎢       1           1           1                  ⎥
⎢         1           1           1                ⎥
⎢           1           1           1              ⎥
⎢             1           1           1            ⎥
⎢               1           1           1          ⎥
⎣                 1           1           1        ⎦
```
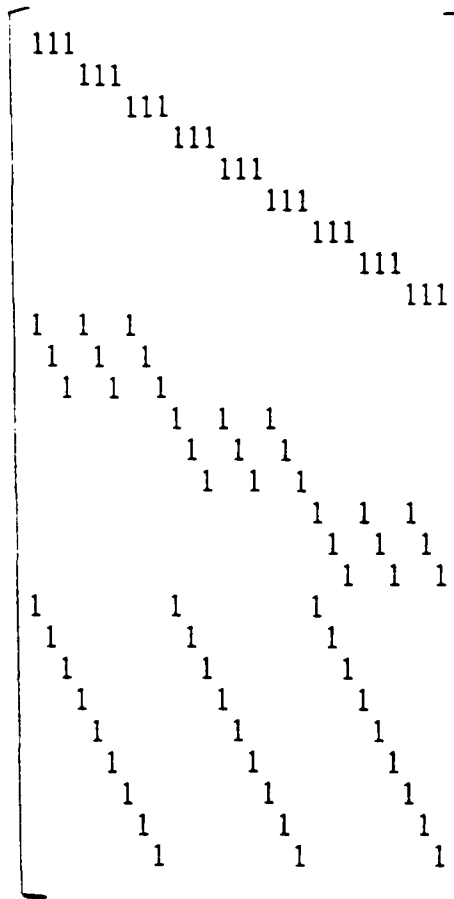
Figure 4

This problem is a special case of the planar 3-dimensional transportation problem. If we view the cube as a plane of n^2 columns of unit cubes, then the constraints of the planar problem require that the sum of the values of variables in each column be exactly one, in each of the three possible orientations. This is illustrated in Figure 5.
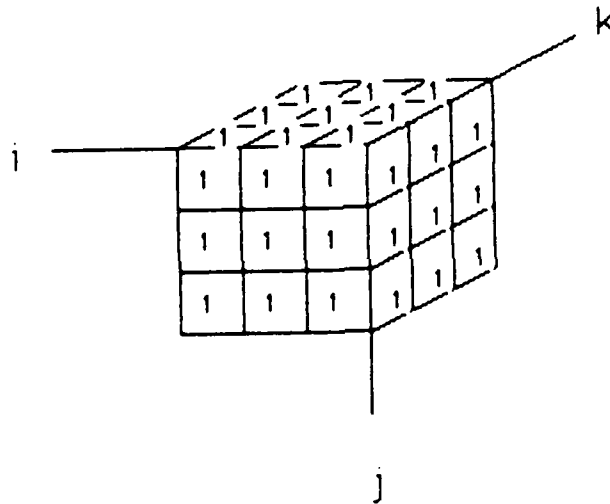

Figure 5

The 3-dimensional transportation problem (TR3) in these and other formulations was first studied by Schell [20]. The literature on this problem includes the references [2,4,5,9,10,12,13,14,15,18,19,20,21]. The original motivation for considering this model was a problem in the transportation of

goods of several types from multiple sources to multiple destinations. Applications of AP3 mentioned in the literature include the following (Pierskalla [18,19]).

- In a rolling mill with $|I|$ soaking pits (temperature stabilizing baths), schedule $|K|$ ingots through the pits so as to minimize idle-time for the rolling mill (the next stage in the process).

- Find a minimum cost schedule of a set of capital investments (e.g., warehouses or plants) in different locations at different times.

- Assign troops to locations over time to maximize a measure of capability.

- Launch a number of satellites in different directions at different altitudes to optimize coverage or minimize cost.

AP3 is known to be an NP-complete problem [11]. Obviously, AP3 is a special case of the *set partitioning problem* (SPP):

$$\max cx \qquad (1)$$

$$\text{s.t. } Bx = e$$

$$x \in \{0,1\}^q$$

where $B = (b_{rs})$ is a matrix of zeroes and ones and e is a vector of ones. The set being partitioned in this case is $R = I \cup J \cup K$, with the rows of B corresponding to the elements $r \in R$. The subsets $s = \{i,j,k\}$ from which the partition is to be selected are those subsets (of cardinality three) which contain one element from each of the sets I, J and K. Each column b^S of B is the *incidence vector* of one such subset (i.e., b^S has a one in each row corresponding to an element in the subset and zeroes elsewhere). The *set packing relaxation* (SP) of (1) is the program:

$$\max cx \qquad (2)$$

$$\text{s.t. } Bx \leq e$$

$x \in \{0,1\}^q.$

For properties of SPP and SP see the survey [3].

Let P_I denote the convex hull of feasible solutions to $AP3_n$, i.e.,

$$P_I = conv\{x \in \{0,1\}^{n^3} | A_n x = e\}$$

The *intersection graph* $G_A = (V,E)$ of a 0-1 matrix A has a node s for every column a^s of A, and an edge (s,t) for every pair of columns a^s, a^t such that $a^s \cdot a^t \neq 0$. The intersection graph G_{A_n} of A_n is the *clique-intersection graph* of $K_{n,n,n}$, i.e., G_{A_n} has a node for every clique (triangle) of $K_{n,n,n}$, and an edge for every pair of triangles that share some node of $K_{n,n,n}$. The graph G_{A_n} for $n=2$ is shown in figure 6.
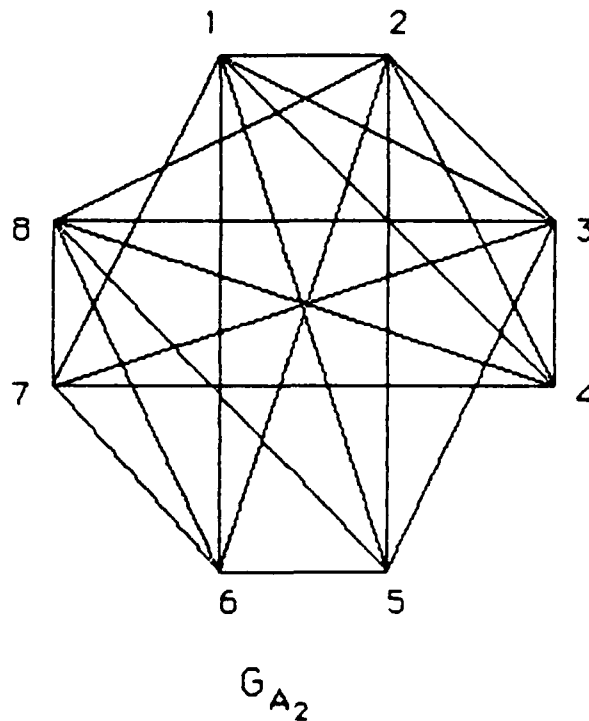

G_{A_2}

Figure 6

Although the 3-index assignment problem has a sizeable literature, no work has been done on describing the polytope P_I. In this paper we apply the tools of polyhedral combinatorics to $AP3_n$ and obtain a partial characterization of the facial structure of P_I. In particular, in section 2 we identify three classes of cliques of the intersection graph of A_n and show that they are exhaustive. These cliques are known to define facets of the polytope

$$P_I = \text{conv}\{x \in \{0,1\}^{n^3} | A_n x \le e\},$$

the set packing relaxation of the set partitioning polytope P_I. In section 3 we show that two of the 3 classes of cliques also define facets of P_I (whereas the cliques in the third class define improper faces), and that these facets are all distinct. Finally, in section 4 we give an $O(n^4)$ procedure for detecting a clique inequality violated by some solution to the linear programming relaxation of P_I, or showing that no such inequality exists.

2. The Cliques of G_A

In this section we identify all the cliques of G_A, the intersection graph of A.

For any subset $V \subseteq S$ of the node set of G_A, we will denote by $<V>$ the subgraph induced by V. For $r \in R$, we will denote by S^r the support of row r of A, i.e., $S^r := \{s \in S | a_{rs} = 1\}$.

Proposition 2.1. For each $r \in R$, the node set S^r induces a clique (of cardinality n^2) in G_A.

Proof. The subgraph $\langle S^r \rangle$ is obviously complete. To see that it is maximal, assume w.l.o.g. that $r \epsilon I$ (an analogous reasoning holds if $r \epsilon J$ or $r \epsilon K$). Now let $s \epsilon S \backslash S^r$ be arbitrarily chosen, and let $s = (i_0, j_0, k_0)$. Since $s \not\epsilon S^r$, $r \neq i_0$; and Since S^r contains all triplets whose first element is r, there exists $t \epsilon S^r$, $t = (r, j, k)$, such that $r \neq i_0$, $j \neq j_0$, $k \neq k_0$. Hence $S^r \cup \{s\}$ does not induce a complete subgraph of G_A; and since this is true of any $s \epsilon S \backslash S^r$, the subgraph of G_A induced by S^r is maximally complete, i.e., a clique. Furthermore, $|S^r| = n^2$ for all $r \epsilon R$. ||

The set of cliques defined by Proposition 2.1 will be called *class 1* and denoted Q_1. Clearly, $|Q_1| = 3n$. In terms of $K_{n,n,n}$, the clique of class 1 corresponding to row r of A contains those nodes of the intersection graph G_A, whose associated triangles in $K_{n,n,n}$ share node r of $K_{n,n,n}$.

Proposition 2.2. For every $s \epsilon S$, let

$$T(s) = \{t \epsilon S \backslash \{s\} \mid a^s \bullet a^t = 2\}.$$

Then the node set $\{s\} \cup T(s)$ induces a clique of size $3n - 2$ in G_A.

Proof. Let $s = (i_0, j_0, k_0)$, and let $t_1, t_2 \epsilon T(s)$ be chosen arbitrarily, with $t_1 \neq t_2$. Since each of t_1 and t_2 contains two of the three elements i_0, j_0, k_0, t_1 and t_2 must have at least one element in common. Hence the node set $\{s\} \cup T(s)$ induces a complete subgraph in G_A. Now let $u \epsilon S \backslash (\{s\} \cup T(s))$. Then the triplet $u = \{i, j, k\}$ contains at most one element of s. If $a^u \bullet a^s = 0$, we are done. Assume now that $a^u \bullet a^s = 1$, with $i = i_0$ (a similar reasoning holds if $j = j_0$ or $k = k_0$). Then $j \neq j_0$ and $k \neq k_0$. Now by definition, $T(s)$ contains some $t = (i_*, j_0, k_0)$ such that $i_* \neq i_0(=i)$. But then $a^u \bullet a^t = 0$, i.e., $\{u\} \cup \{s\} \cup T(s)$ does not define a complete subgraph of G_A. Since the choice of u was arbitrary, the subgraph defined by $\{s\} \cup T(s)$ is maximal complete.

For each $s \in S$ and for each of the three pairs of the triplet $s = (i_0, j_0, k_0)$, there are $n-1$ other triplets in S containing the same pair; hence $|T(s)| = 3(n-1)$, and thus $\{s\} \cup T(s)$ has $3n-2$ elements. ||

The set of cliques defined in Proposition 2.2 will be called <u>class 2</u> and denoted Q_2. There is exactly one clique of class 2 for every column of A, and there is no double counting; hence $|Q_2| = n^3$. In terms of $K_{n,n,n}$, the clique of class 2 corresponding to column $s = (i_0, j_0, k_0)$ of A contains the node of G_A corresponding to the clique (i_0, j_0, k_0) of $K_{n,n,n}$, along with the $3(n-1)$ nodes of G_A corresponding to those cliques of $K_{n,n,n}$ that share an edge (a pair of nodes) with the clique (i_0, j_0, k_0).

<u>Proposition 2.3.</u> For every ordered pair $s,t \in S$ such that $a^s \cdot a^t = 0$, let $t_1, t_2, t_3 \in S \setminus \{s,t\}$ be the (uniquely defined) triplets such that
$$a^s \cdot a^{t_i} = 1, \quad a^t \cdot a^{t_i} = 2, \qquad i = 1,2,3.$$

Then the node set $\{s, t_1, t_2, t_3\}$ induces a (4-)clique in G_A.

<u>Proof.</u> Let $s,t \in S$, with $a^s \cdot a^t = 0$, and let $s = (i_s, j_s, k_s)$, $t = (i_t, j_t, k_t)$. Then $t_1 = (i_s, j_t, k_t)$, $t_2 = (i_t, j_s, k_t)$ and $t_3 = (i_t, j_t, k_s)$ are the only 3 triplets in $S \setminus \{s,t\}$ that satisfy the requirements of the Proposition , i.e., they exist and are unique. Further, $a^s \cdot a^{t_i} = 1$ for $i=1,2,3$ and $a^{t_i} \cdot a^{t_j} = 2$ for all $i,j \in \{1,2,3\}$; hence $\{s, t_1, t_2, t_3\}$ induces a complete subgraph in G_A. To see that this subgraph is maximal, note that any triplet $u \in S \setminus \{s\}$ that contains an element of s, either contains two elements of t (and hence is identical to one of the triplets t_1, t_2 or t_3), or else contains at most one element of t. But then $a^u \cdot a^{t_i} = 0$, where $t_i \in \{t_1, t_2, t_3\}$ is the triplet containing those two elements of t not contained in u (besides the element of s). Thus $\{s, t_1, t_2, t_3\}$ induces a maximal complete subgraph, hence a 4-clique in G_A. ||

The set of cliques described in Propositions 2.3 will be called class 3 and denoted Q_3. In terms of $K_{n,n,n}$, every class 3 clique of G_A is associated with an ordered pair (s,t) of disjoint triangles of $K_{n,n,n}$, and its node set contains (a) the node of G_A corresponding to the triangle s, and (b) the 3 nodes of G_A corresponding to those triangles t_1, t_2, t_3 of $K_{n,n,n}$ that share 1 node with s and 2 nodes with t.

As to the cardinality of Q_3, every ordered pair (s,t) such that $a^s \cdot a^t = 0$ gives rise to a clique of class 3. Since $|S| = n^3$ and for every $s \in S$ that are $(n-1)^3$ indices $t \in S$ such that $a^s \cdot a^t = 0$, the number of ordered pairs (s,t) with $a^s \cdot a^t = 0$ is $n^3(n-1)^3$.

To determine the number of cliques of class 3 we also need to know how many different ordered pairs give rise to the same clique. Let $s = (i_s, j_s, k_s)$, $t = (i_t, j_t, k_t)$, $t_1 = (i_s, j_t, k_t)$, $t_2 = (i_t, j_s, k_t)$, $t_3 = (i_t, j_t, k_s)$, and denote by $C(s,t)$ the node set of the clique (of class 3) corresponding to the ordered pair (s,t), i.e. let $C(s,t) := \{s, t_1, t_2, t_3\}$. Further, let $\bar{t}_1 = (i_t, j_s, k_s)$, $\bar{t}_2 = (i_s, j_t, k_s)$, $\bar{t}_3 = (i_s, j_s, k_t)$. Then we have

<u>Proposition 2.4.</u> $C(s,t) = C(t_i, \bar{t}_i)$ for $i = 1,2,3$.

<u>Proof.</u> Consider the ordered pair $(t_1, \bar{t}_1)$. From the definitions, the 4 triplets of the set $C(t_1, \bar{t}_1)$ are $(i_s, j_t, k_t) = t_1$, $(i_s, j_s, k_s) = s$, $(i_t, j_t, k_s) = t_3$, and $(i_t, j_s, k_t) = t_2$; thus $C(t_1, \bar{t}_1) = C(s,t)$. By symmetry, $C(t_i, \bar{t}_i) = C(s,t)$ for $i = 2,3$. ||

<u>Corollary 2.5.</u> The number of cliques of class 3 is $n^3(n-1)^3/4$.

<u>Proof.</u> Every clique of class 3 arises from 4 distinct ordered pairs, and the number of the latter is $n^3(n-1)^3$. ||

Next we show that G_A has no other cliques than the ones described above. But first we need a property of G_A.

Proposition 2.6. G_A is regular of degree $3n(n-1)$ and has $\frac{3}{2} n^4(n-1)$ edges.

Proof. Let a^s be an arbitrary column of A. There are $(n-1)^3$ columns a^t of A such that $a^s \cdot a^t = 0$, hence there are $n^3-1-(n-1)^3 = 3n(n-1)$ columns a^u of A such that $a^s \cdot a^u \neq 0$. Thus the degree of node s in G_A is $3n(n-1)$, and by symmetry this is true of all $s \in S$. Since the number of edges of a graph is one half of the sum of the degrees of its nodes, G_A has $\frac{1}{2} \times n^3 \times 3n(n-1) = \frac{3}{2} n^4(n-1)$ edges. ||

Let Q denote the set of cliques of G_A.

Theorem 2.7. The only cliques of G_A are those of classes 1, 2 and 3; i.e., $Q = Q_1 \cup Q_2 \cup Q_3$.

Proof. We will use induction on n. For $n = 2$, the statement is found to be true by listing all the cliques of G_A (see Fig. 4). In fact, because of symmetry, it is sufficient to list the cliques containing a given node, say 1, and they are $\{1,2,3,4\}$, $\{1,2,5,6\}$, $\{1,3,5,7\}$ (of class 1), $\{1,2,3,5\}$, $\{1,2,4,6\}$, $\{1,3,4,7\}$, $\{1,5,6,7\}$ (of class 2), and $\{1,4,6,7\}$ (of class 3).

Suppose now that the statement is true for $n = 2,\ldots,k$ and let $n = k+1 \geq 3$. Consider the relationship between $G_{A_{n-1}}$ and G_{A_n}. Note first that $K_{n,n,n}$ is obtained from $K_{n-1,n-1,n-1}$ by (a) adding three nodes i_*, j_*, k_*, to the three sets I, J, K, respectively, so that

$$I_n = I_{n-1} \cup \{i_*\}, \ J_n = J_{n-1} \cup \{j_*\} \text{ and } K_n = K_{n-1} \cup \{k_*\};$$

and (b) adding new edges (i_*, ℓ) for all $\ell \in J_n \cup K_n$, (j_*, ℓ) for all $\ell \in I_n \cup K_n$, and (k_*, ℓ) for all $\ell \in I_n \cup J_n$.

This creates $3n(n-1)+1$ new triangles, namely:

(i) one new triangle (i_*, j_*, k_*) that shares no node with any triangle of $K_{n-1,n-1,n-1}$;

(ii) $3(n-1)$ new triangles of the form (ℓ, j_*, k_*) for all $\ell \in I_{n-1}$,

(i_*,ℓ,k_*) for all $\ell \in J_{n-1}$, and (i_*,j_*,ℓ) for all $\ell \in K_{n-1}$; and

(iii) $3(n-1)^2$ new triangles of the form (p,q,k_*) for all

$(p,q) \in I_{n-1} \times J_{n-1}$, (p,j_*,q) for all $(p,q) \in I_{n-1} \times K_{n-1}$, and

(i_*,p,q) for all $(p,q) \in J_{n-1} \times K_{n-1}$.

In terms of the coefficient matrix of AP3, A_n is obtained from A_{n-1} by adding 3 new rows, one to each of the sets I_{n-1}, J_{n-1}, K_{n-1}, with zero entries in the positions indexed by S_{n-1}; and adding $3n(n-1)+1$ new columns (of dimension $|R_n| = |R_{n-1}|+3$), with supports corresponding to the new triangles of $K_{n,n,n}$ described under (iii) above.

It then follows that G_A is obtained from $G_{A_{n-1}}$ by adding $3n(n-1)+1$ new nodes corresponding to the triangles described under (i), (ii), (iii), and a new edge for every pair of nodes (s,t) of G_{A_n} such that (a) at most one of s and t is a node of $G_{A_{n-1}}$, and (b) the two triangles of $K_{n,n,n}$ corresponding to s and t have at least one node (of $K_{n,n,n}$) in common. It also follows that $G_{A_{n-1}}$ is an induced subgraph of G_{A_n}.

Consider now the node set C of an arbitrary clique of G_{A_n}. The restriction $\bar{C}$ of C to G_A induces a clique of $G_{A_{n-1}}$, hence by the induction hypothesis $<\bar{C}>$ belongs to one of the three classes described in Propositions 2.1, 2.2, 2.3. With an argument analogous to the one used in the proof of the corresponding Proposition to show that a complete subgraph of the given class is maximal, we will show for each of the 3 cases that $<C>$ belongs to the same class as $<\bar{C}>$. Since $<C>$ was chosen arbitrarily, this will prove that the three classes of cliques are exhaustive for $G_{A_n} = G_{A_{k+1}}$, thus completing the induction.

Suppose first that $<\bar{C}> \in Q_1$. Then $\bar{C}$ is of the form $\bar{C} = S_{n-1}^i = \{j \in S_{n-1} | a_{ij} = 1\}$ for some $i \in R_{n-1}$. W.l.o.g. we may assume that $i \in I_{n-1}$ (an analogous reasoning holds if $i \in J_{n-1}$, or $i \in K_{n-1}$).

Now suppose $<C> \not\subseteq Q_1$. Then $<C>$ has a node $s = \{i_0, j_0, k_0\}$ such that $i_0 \neq i$. Since S_{n-1}^i contains all the triplets of S_{n-1} whose first element is i, it certainly contains some $t = (i, j, k)$ such that $j \neq j_0$ and $k \neq k_0$. But then $a^s \bullet a^t = 0$ (where a^s and a^t are columns of A_n), hence we obtain the contradiction that $<C>$ is not a clique.

Suppose next that $<C> \in Q_2$. Then C is of the form $\{s\} \cup T(s)$, where $s \in S_{n-1}$ and $T(s) = \{t \in S_{n-1} \setminus \{s\} \mid a^s \bullet a^t = 2\}$ (a^s and a^t being columns of A_{n-1}). Let $s = (i_0, j_0, k_0)$. Now suppose $<C> \not\in Q_2$. Then $<C>$ has a node $u = (i, j, k)$, such that $a^s \bullet a^u \leq 1$ (where a^s and a^u are columns of A_n). Since $<C>$ is a clique, $a^s \bullet a^u \neq 0$, hence $a^s \bullet a^u = 1$. W.l.o.g., assume that the common element of the triplets s and u is i_0 (a similar reasoning holds if the common element is j_0 or k_0). Then $j \neq j_0$ and $k \neq k_0$. By definition, $T(s)$ contains some $t = (i_*, j_0, k_0)$ such that $i_* \neq i_0 (=i)$, and $t \in T(s)$ implies $t \in C$. But then $a^u \bullet a^t = 0$, contradicting the assumption that $<C>$ is a clique.

Finally, suppose $<C> \in Q_3$. Then C is of the form $\{s, t_1, t_2, t_3\}$, where $s, t_1, t_2, t_3 \in S_{n-1}$ are distinct and such that

$$a^s \bullet a^{t_i} = 1, \qquad a^t \bullet a^{t_i} = 2, \qquad i = 1, 2, 3,$$

for some $t \in S_{n-1}$ such that $a^s \bullet a^t = 0$. (Here a^s, a^t and all a^{t_i} are columns of A_{n-1}). Let $s = (i_s, j_s, k_s)$, $t = (i_t, j_t, k_t)$, and $t_1 = (i_s, j_t, k_t)$, $t_2 = (i_t, j_s, k_t)$, $t_3 = (i_t, j_t, k_s)$. Now suppose $<C> \not\in Q_3$. Then $<C>$ has a node $u \in S_n \setminus S_{n-1}$, say $u = (i, j, k)$, such that $a^u \bullet a^s > 0$, $a^u \bullet a^{t_i} > 0$, $i = 1, 2, 3$ (here a^u, a^s and all a^{t_i} are columns of A_n). Since $u \not\in S_{n-1}$, at least one of the elements i, j, k is not contained in any of the 4 triplets s, t_1, t_2, t_3. W.l.o.g. assume k is such an element. Then each of s, t_1, t_2, t_3 must contain either i or j. There are four cases: $i = i_s$, $j = j_s$; $i = i_t$, $j = j_t$; $i = i_s$, $j = j_t$; and $i = i_t$, $j = j_s$. In each case at least one of the four

triplets s, t_1, t_2, t_3 does not contain either i or j, which contradicts the assumption that <C> is a clique.

Thus every clique of $G_{A_n} = G_{A_{k+1}}$ belongs to either Q_1, or Q_2, or Q_3, and the induction is complete. ||

3. Facets of P_I Induced by Cliques of G_A.

If C is the vertex set of a clique of G_A, then obviously every $x \in P$ satisfies the inequality

$$(3.1) \qquad \sum_{s \in C} x_s \leq 1.$$

Such inequalities are known to define facets of P_I, the set packing polytope associated with P_I[17]; but Since P_I itself is a face of P_I, it is an open question whether an inequality (3.1) also defines a facet of P_I. In this section we answer this question exhaustively.

First, some definitions and basic concepts. For any polyhedron P, let dim P denote the dimension of P (defined as the dimension of the affine hull of P, i.e. of the smallest subspace containing P). An inequality $\pi x \leq \pi_0$ is said to define a *facet* of P, if it is satisfied by every $x \in P$ and the polyhedron $P^\pi := \{x \in P | \pi x = \pi_0\}$ has dimension dim P-1. If $\pi x = \pi_0$ for all $x \in P$, the inequality $\pi x \leq \pi_0$ is said to define an *improper face* of P. In this case of course dim P^π = dim P. To show that $\pi x \leq \pi_0$ does not define an improper face, it is sufficient to exhibit a point $x \in P$ such that $\pi x < \pi_0$. Once this is ascertained to be the case, dim $P^\pi \leq$ dim P-1, since (a) dim P is the number of variables in the system defining P, minus the rank of the equality system of P (i.e. of the system of linear *equations* satisfied

by all $x \in P$); and (b) the addition of the equation $\pi x = \pi_0$, not implied by the system defining P, increases the rank of the equality system by at least 1. Thus showing that $\pi x \leq \pi_0$ defines a facet of P essentially amounts to showing that the dimension of P^{π}, known to be bounded by dim $P-1$, is actually equal to this bound. The most commonly used procedure for doing this is to exhibit dim P affinely independent points $x \in P^{\pi}$. Another way of doing it is to show that the addition of $\pi x = \pi_0$ to the constraints defining P increases the rank of the equality system of P by exactly one; in other words, that *any* equation satisfied by all $x \in P^{\pi}$ is a linear combination of the equations in the system defining P^{π}. In this paper we will take the latter approach, and will use it also to establish the dimension of P_I itself. We will implement this approach via a technique similar to that used by Maurras [16], as well as by Cornuejols and Pulleyblank [5], (see also Cornuejols and Thizy [6]).

We first establish the dimension of P_I.

Let P denote the feasible set of the linear programming relaxation of P_I, i.e.

$$P = \{x \in R^{n^3} | Ax = e, x \geq 0\}.$$

Lemma 3.1. The rank of the system $Ax = e$ is $3n-2$.

Proof. The rank of $Ax = e$ is at most $3n-2$, since equation $2n$ is the sum of the first n equations, minus the sum of equations $n+1,\ldots,2n-1$; and equation $3n$ is the sum of the first n equations minus the sum of equations $2n+1,\ldots,3n-2$. On the other hand, the rank of $Ax = e$ is at least $3n-2$, since we can exhibit $3n-2$ affinely independent columns of A. Consider the three sets of columns indexed by the following triplets:

$$(2,1,1), (3,1,1), \ldots, (n,1,1);$$

$$(1,2,1), (1,3,1), \ldots, (1,n,1);$$

$$(1,1,1), (1,1,2), \ldots, (1,1,n)$$

The first two sets contain n-1 columns each, the last one contains n columns. The matrix formed by these columns (in the order of their listing), after deletion of the first row of set I and the first row of set J, becomes a square lower triangular (hence nonsingular) matrix of order 3n-2, with each diagonal element equal to 1 and all elements above the diagonal equal to 0. ||

Corollary 3.2. dim $P = n^3 - 3n + 2$.

Proof. The dimension of P is the number of variables in its defining system (n^3), minus the rank of its equality system $Ax = e$ (3n-2). ||

We are interested in dim P_I. Since $P_I \subset P$, dim $P_I \leq n^3 - 3n + 2$, and strict inequality holds if and only if there exists an equation $\alpha x = \alpha_0$ satisfied by all $x \in P_I$, that is not implied by (not a linear combination of) the equations $Ax = e$. We will show that no such equation exists.

Theorem 3.3. Suppose every $x \in P_I$ satisfies $\alpha x = \alpha_0$ for some $\alpha \in R^{n^3}$, $\alpha_0 \in R$. Then there exist scalars $\lambda_i, \forall i \in I, \mu_j, \forall j \in J,$ and $\nu_k, \forall k \in K$, such that

$$\alpha_{ijk} = \lambda_i + \mu_j + \nu_k, \quad \forall (i,j,k) \in I \times J \times K$$

and

$$\alpha_0 = \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k.$$

Proof. Let $x \in P_I$, $x \in \{0,1\}^{n^3}$ be such that $x_{i_1 j_1 k_1} = x_{i_3 j_2 k_2} = 1$, and let $x' \in \{0,1\}^{n^3}$ be defined by

$$x'_{i_3 j_1 k_1} = x'_{i_1 j_2 k_2} = 1,$$

$$x'_{i_1 j_1 k_1} = x'_{i_3 j_2 k_2} = 0,$$

and $x'_{ijk} = x_{ijk}$ for all other (i,j,k). It is easy to see that $x \in P_I$ implies $x' \in P_I$, and Since $\alpha x = \alpha x' (=\alpha_0)$ by assumption, we have

$$(3.2) \qquad \alpha_{i_1 j_1 k_1} + \alpha_{i_3 j_2 k_2} = \alpha_{i_3 j_1 k_1} + \alpha_{i_1 j_2 k_2} \; .$$

Next, let $\bar{x} \in P_I$, $\bar{x} \in \{0,1\}^{n^3}$ be such that $\bar{x}_{i_2 j_1 k_1} = \bar{x}_{i_3 j_2 k_2} = 1$, and let $\bar{x}' \in \{0,1\}^{n^3}$ be defined by

$$\bar{x}'_{i_3 j_1 k_1} = \bar{x}'_{i_2 j_2 k_2} = 1,$$

$$\bar{x}'_{i_2 j_1 k_1} = \bar{x}'_{i_3 j_2 k_2} = 0,$$

and $\bar{x}'_{ijk} = \bar{x}_{ijk}$ for all other (i,j,k). Again, $\bar{x} \in P_I$ implies $\bar{x}' \in P_I$, and since $\alpha\bar{x} = \alpha\bar{x}'$, we have

$$(3.3) \qquad \alpha_{i_2 j_1 k_1} + \alpha_{i_3 j_2 k_2} = \alpha_{i_3 j_1 k_1} + \alpha_{i_2 j_2 k_2} \; .$$

Subtracting (3.3) from (3.2) gives

$$(3.4) \qquad \alpha_{i_1 j_1 k_1} - \alpha_{i_2 j_1 k_1} = \alpha_{i_1 j_2 k_2} - \alpha_{i_2 j_2 k_2}$$

If in the definition of x, x' and of $\bar{x}$, $\bar{x}'$ we replace the pairs (j_2,k_2) by arbitrary $(j,k) \in J \times K$, we obtain (3.4) with (j_2,k_2) on the right hand side replaced by (j,k). Thus the meaning of (3.4) is that two components of α whose index-triplets differ only in their first element, differ by an amount depending only on the values of those first elements, i.e.

$$(3.5) \qquad \alpha_{i_1 jk} - \alpha_{i_2 jk} = \text{const.}, \quad \forall (j,k) \in J \times K.$$

By symmetry we have equations analogous to (3.5) for components of α whose index-triplets differ only in their second or only in their third element.

Now let us denote

$$(3.6) \quad \begin{aligned} \beta_{i_1 i_2} &= \alpha_{i_1 jk} - \alpha_{i_2 jk} \\ \gamma_{j_1 j_2} &= \alpha_{i j_1 k} - \alpha_{i j_2 k} \\ \delta_{k_1 k_2} &= \alpha_{i j k_1} - \alpha_{i j k_2} \end{aligned}$$

Then we have

$$\alpha_{ijk} - \alpha_{i_1 j_1 k_1} = \alpha_{ijk} - \alpha_{i_1 jk} + \alpha_{i_1 jk} - \alpha_{i_1 j_1 k} + \alpha_{i_1 j_1 k} - \alpha_{i_1 j_1 k_1}$$

$$= \beta_{i i_1} + \gamma_{j j_1} + \delta_{k k_1},$$

or

$$(3.7) \quad \alpha_{ijk} = \beta_{i i_1} + \gamma_{j j_1} + \delta_{k k_1} + \alpha_{i_1 j_1 k_1}, \quad \forall (i,j,k) \in I \times J \times K.$$

Next we fix μ_{j_1} and ν_{k_1} arbitrarily, we let $\lambda_{i_1} = \alpha_{i_1 j_1 k_1} - \mu_{j_1} - \nu_{k_1}$, and define

$$(3.8) \quad \begin{aligned} \lambda_i &= \lambda_{i_1} + \beta_{i i_1}, & i \in I \setminus \{i_1\}, \\ \mu_j &= \mu_{j_1} + \gamma_{j j_1}, & j \in J \setminus \{j_1\}, \\ \nu_k &= \nu_{k_1} + \delta_{k k_1}, & k \in K \setminus \{k_1\}. \end{aligned}$$

Then

$$(3.9) \quad \alpha_{ijk} = \lambda_i + \mu_j + \nu_k, \quad \forall (i,j,k) \in I \times J \times K.$$

Further, let $\hat{x}$ be defined by

$$\hat{x}_{ijk} = \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otherwise.} \end{cases}$$

Then $\bar{x} \in P_I$, hence $\alpha\bar{x} = \alpha_0$, or

$$\alpha_0 = \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k. \quad ||$$

Corollary 3.4. dim $P_I = n^3 - 3n + 2$.

Proof. From Theorem 3.3, the smallest affine subspace containing P_I is the one defined by the system $Ax = e$; the dimension of P_I is therefore the same as that of P. $||$

Next we turn to the constraints defining P and ask the question, which ones among these define facets of P_I.

Theorem 3.5. Every inequality $x_s \geq 0$ for some $s \in S$ defines a facet of P_I.

Proof. The statement is true if and only if the polytope $P_I^S = \{x \in P_I | x_s = 0\}$ has dimension dim $P_I - 1 = n^3 - 3n + 1$. Clearly, dim $P_I^S \leq n^3 - 1 - r$, where r is the rank of the system $A^S x = e$, and A^S is the matrix obtained from A by removing the column a^s. The rank of A^S is easily seen to be the same as the rank of A, i.e. $r = 3n-2$. This is immediate in the case when a^s is not among those columns used in the proof of Lemma 3.1, and follows by symmetry for the other case. Hence the dimension of P_I^S is at most $n^3 - 3n + 1$. To show that this bound is actually attained, one can use the same argument as in the proof of Theorem 3.3 to show that any equation $\alpha x = \alpha_0$ (other than $x_s = 0$) satisfied by every $x \in P_I^S$ is a linear combination of the equations $A^S x = e$. The argument goes through essentially unchanged. $||$

As to the equations of the system $Ax = e$, each of them defines an improper face of P_I (i.e. is satisfied by every $x \in P_I$).

We now turn to the inequalities (3.1) defined by the cliques of G_A.

Each clique of class 1 induces an inequality whose left hand side coefficient vector is one of the rows of A. Hence each such inequality is satisfied with equality by every $x \in P_I$ and therefore defines an improper face of P_I.

Next we consider the inequalities (3.1) induced by the cliques of class 2. Each clique in this class is defined relative to some index (triplet) $s \in S$, and has a node set of the form $\{s\} \cup T(s)$ (see Proposition 2.2). Let $C(s)$ denote the node set of the clique of class 2 defined relative to s.

Theorem 3.6. For $n \geq 3$, the inequality

$$(3.10) \qquad \sum_{t \in C(s)} x_t \leq 1$$

defines a facet of P_I for every $s \in S$.

Proof. Let $P_I^{C(s)} = \{x \in P_I \mid \sum_{t \in C(s)} x_t = 1\}$ and let $n \geq 3$. We will show that $\dim P_I^{C(s)} \neq P_I$. Let $s = (i_0, j_0, k_0)$; then every triplet in $C(s) \setminus \{s\}$ contains two of the three elements of s. Hence any $x \in P_I$ whose support includes (i_0, j', k'), (i', j_0, k'') and (i'', j'', k_0), for any i', $i'' \neq i_0$, j', $j'' \neq j_0$ and k', $k'' \neq k_0$, satisfies $\sum_{t \in C(s)} x_t = 0 < 1$. Thus (3.9) does not define

an improper face of P_I, i.e., $\dim P_I^{C(s)} \leq P_I - 1$.

To show that this last inequality holds as equality, we prove that an arbitrary equation $ax = a_0$ satisfied by all $x \in P_I^{C(s)}$ is a linear combination of the equations of the system $Ax = e$ and

$$(3.11) \qquad \sum_{t \in C(s)} x_t = 1.$$

Let $x \in C(s)$, $x \in \{0,1\} \subset R^{n^3}$, be such that $x_{i_1 j_1 k_1} = x_{i_3 j_2 k_2} = 1$, where (i_1, j_1, k_1), $(i_3, j_2, k_2) \notin C(s)$. Since $n \geq 3$, there exist at least 3 disjoint triplets (i, j, k) such that $x_{ijk} = 1$, and at most one of these can belong to

$C(s)$; hence the vector $x \in P^{C(s)}$ postulated here exists. Note that x is of the same form as the vector x used in the proof of Theorem 3.3, and hence the same reasoning can be used to derive the relation

$$\alpha_{i_1 jk} - \alpha_{i_2 jk} = \text{const.}, \quad \forall (j,k) \in J \times K : (i_1, j, k), (i_2, j, k) \notin C(s)$$

for all $i_1, i_2 \in I$, analogous to (3.5). Using the notation (3.6), we then obtain

$$\alpha_{ijk} = \beta_{ii_1} + \gamma_{jj_1} + \delta_{kk_1} + \alpha_{i_1 j_1 k_1}, \quad \forall (i,j,k) \in S \setminus C(s),$$

and defining λ_i, $i \in I$, μ_j, $j \in J$, ν_k, $k \in K$ as in (3.8) we get the relation

(3.12) $\quad \alpha_{ijk} = \lambda_i + \mu_j + \nu_k, \quad \forall (i,j,k) \in S \setminus C(s)$

analogous to (3.9).

Now let us define

$$\pi_i^I = \alpha_{ij_o k_o} - \lambda_i - \mu_{j_o} - \nu_{k_o}, \qquad \forall \, i \in I$$

(3.13) $\quad \pi_j^J = \alpha_{i_o j k_o} - \lambda_{i_o} - \mu_j - \nu_{k_o}, \qquad \forall \, j \in J$

$$\pi_k^K = \alpha_{i_o j_o k} - \lambda_{i_o} - \mu_{j_o} - \nu_k, \qquad \forall \, k \in K$$

where $(i_o, j_o, k_o) = s$.

Since $t \in C(s)$ if and only if t is either of the form (i, j_o, k_o), or (i_o, j, k_o), or (i_o, j_o, k), (3.13) defines α_{ijk} for all $(i,j,k) \in C(s)$. We will show that $\pi_i^I = \pi_j^J = \pi_k^K$, $\forall \, i \in I, j \in J, k \in K$. From (3.13) it is easily seen that $\pi_{i_o}^I = \pi_{j_o}^J = \pi_{k_o}^K$; hence, denoting $\pi_{C(s)} := \pi_{i_o}^I \, (= \pi_{j_o}^J = \pi_{k_o}^K)$, it is

sufficient to show that $\pi_i^I = \pi_{C(s)}$, $\forall \, i \in I$: it then follows by symmetry that $\pi_j^J = \pi_k^K = \pi_{C(s)}$, $\forall \, j \in J, k \in K$.

Let $\bar{x} \in P_I^{C(s)}$, $\bar{x} \in \{0,1\}^{n^3}$, be such that $\bar{x}_{i_o j_o k_o} = \bar{x}_{i'j'k'} = 1$, where

$(i',j',k') \notin C(s)$ is chosen arbitrarily; and let $\bar{x}' \in \{0,1\}^{n^3}$ be defined by

$$\bar{x}'_{i'j_o k_o} = \bar{x}'_{i_o j'k'} = 1,$$

$$\bar{x}'_{i_o j_o k_o} = \bar{x}'_{i'j'k'} = 0,$$

and $\bar{x}'_{ijk} = \bar{x}_{ijk}$ for all other (i,j,k). Then from $\bar{x} \in P_I^{C(s)}$ it follows that $\bar{x}' \in P_I^{C(s)}$, and from $\alpha\bar{x} = \alpha\bar{x}' (=\alpha_0)$ we obtain

$$\alpha_{i_o j_o k_o} + \alpha_{i'j'k'} = \alpha_{i'j_o k_o} + \alpha_{i_o j'k'}.$$

Substituting for $\alpha_{i_o j_o k_o}$, $\alpha_{i'j_o k_o}$ from (3.13) and $\alpha_{i_o j'k'}$, $\alpha_{i'j'k'}$ from

(3.12) (since $i_o,j_o,k_o),(i',j_o,k_o) \in C(s)$ and (i_o,j',k'), $(i',j',k') \notin C(s)$)

we obtain

$$\pi_{i_o}^I + \lambda_{i_o} + \mu_{j_o} + \nu_{k_o} + \lambda_{i'} + \mu_{j'} + \nu_{k'} = \pi_{i'}^I + \lambda_{i'} + \mu_{j_o} + \nu_{k_o} + \lambda_{i_o} + \mu_{j'} + \nu_{k'} \text{ or }$$

$\pi_{i_o}^I = \pi_{i'}^I$; and since i' was chosen arbitrarily (subject to the condition

that $(i',j',k') \notin C(s))$, this proves that $\pi_i^I = \pi_{i_o}^I = \pi_{C(s)}, \forall i \in I$.
Furthermore, as mentioned above, $\pi_j^J = \pi_k^K = \pi_{C(s)}, \forall j \in J, k \in K$ follows by
symmetry.

We have thus proved that

$$\alpha_{ijk} = \begin{cases} \lambda_i + \mu_j + \nu_k + \pi_{C(s)}, & \forall (i,j,k) \in C(s) \\ \lambda_i + \mu_j + \nu_k, & \forall (i,j,k) \in S \backslash C(s) \end{cases}$$

Now let $\ell \in \{1,1\}^{n^3}$ be defined by

$$\ell_{ijk} = \begin{cases} 1 & \text{if } i=i_0 + \varepsilon, \; j=j_0 + \varepsilon, \; k=k_0 + \varepsilon \pmod{n} \text{ for } \varepsilon=0,1,\ldots,n-1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\ell \in P_I^{C(s)}$, hence $\alpha\hat{x} = \alpha_0$, i.e.,

$$\alpha_0 = \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k + \pi_{C(s)}.$$

This proves that the equation $\alpha x = \alpha_0$ is a linear combination of the equations of the system defining $P_I^{C(s)}$; hence $\dim P_I^{C(s)} = P_I - 1$ and thus the inequality (3.10) defines a facet of P_I. $||$

Finally, we turn to the inequalities (3.1) induced by cliques of class 3. Remember that each clique in this class is defined relative to an ordered pair (s,t) of disjoint triplets, and has a node set of the form $\{s,t_1,t_2,t_3\}$, where each t_i, $i = 1,2,3$, contains one element of s and two elements of t (see Proposition 2.3). Let $C(s,t)$ denote the node set of the clique of class 3 defined relative to the ordered pair (s,t).

<u>Theorem 3.7</u> For $n \geq 3$, the inequality

$$(3.14) \qquad \sum_{u \in C(s,t)} x_u \leq 1$$

defines a facet of P_I for all $s,t \in S$.

<u>Proof.</u> The proof of Theorem 3.6 can be used to show that $\dim P_I^{C(s,t)} \leq P_I-1$, and to derive the relation

$$(3.15) \qquad \alpha_{ijk} = \lambda_i + \mu_j + \gamma_k, \; \forall \; (i,j,k) \in S \backslash C(s,t).$$

analogous to (3.12).

Now let $s = (i_s, j_s, k_s)$, $t = (i_t, j_t, k_t)$, and define

$$\pi_s = \alpha_{i_s j_s k_s} - \lambda_{i_s} - \mu_{j_s} - \nu_{k_s}$$

$$\pi_{t_1} = \alpha_{i_s j_t k_t} - \lambda_{i_s} - \mu_{j_t} - \nu_{k_t}$$

(3.16)

$$\pi_{t_2} = \alpha_{i_t j_s k_t} - \lambda_{i_t} - \mu_{j_s} - \nu_{k_t}$$

$$\pi_{t_3} = \alpha_{i_t j_t k_s} - \lambda_{i_t} - \mu_{j_t} - \nu_{k_s}$$

To show that $\pi_{t_i} = \pi_s$, $i = 1,2,3$, consider $\bar{x} \in \{0,1\}^{n^3}$, $\bar{x} \in P_I^{C}(s,t)$ such that $\bar{x}_{i_s j_s k_s} = \bar{x}_{i_t j_t k_t} = 1$, and define $\bar{x}'$ by

$$\bar{x}'_{i_t j_s k_s} = \bar{x}'_{i_s j_t k_t} = 1,$$

$$\bar{x}'_{i_s j_s k_s} = \bar{x}'_{i_t j_t k_t} = 0,$$

and $\bar{x}'_{ijk} = \bar{x}_{ijk}$ for all other (i,j,k). Then $\bar{x}' \in P_I^{C}(s,t)$ follows from $\bar{x} \in P_I^{C}(s,t)$, and $\alpha \bar{x} = \alpha \bar{x}'$ implies

$$\alpha_{i_s j_s k_s} + \alpha_{i_t j_t k_t} = \alpha_{i_t j_s k_s} + \alpha_{i_s j_t k_t}.$$

Since (i_s, j_s, k_s), $(i_s, j_t, k_t) \in C(s,t)$
and $(i_s, j_s, k_t), (i_t, j_s, k_s) \notin C(s,t)$, we have (from (3.15) and (3.16))

$$\pi_s + \lambda_{i_s} + \mu_{j_s} + \nu_{k_s} + \lambda_{i_t} + \mu_{j_t} + \nu_{k_t} = \pi_{t_1} + \lambda_{i_s} + \mu_{j_t} + \nu_{k_t} + \lambda_{i_t} + \mu_{j_s} + \nu_{k_s}$$

or $\pi_{t_1} = \pi_s$. By symmetry, $\pi_{t_2} = \pi_s = \pi_{t_3}$, and we are done.$\|$

<u>Theorem 3.8</u> The inequalities (3.1) induced by distinct cliques define distinct facets when $n \geq 3$.

To prove this theorem, we require the following auxiliary result:

<u>Lemma 3.9</u> For $n \geq 3$, there is a feasible solution x with $x_s = x_t = x_r = 1$ and a pair of cliques with vertex sets C_1, C_2, such that 1. $s \in C_1 \backslash C_2$; 2. $t \in C_2 \backslash C_1$; 3. $r \in S \backslash (C_2 \cup C_1)$; 4. if $C_2 = C(t_0)$ then $t \neq t_0$ and $t_0 \cap r = \emptyset$; and if $C_2 = C(t, r_0)$ then $r \neq r_0$.

    <u>Proof.</u>   (i) Suppose $C_1 = C(s_0)$, $C_2 = C(t_0)$ where $s_0 = (i_{s_0}, j_{s_0}, k_{s_0})$, $t_0 = (i_{t_0}, j_{t_0}, k_{t_0})$. If $t_0 \in C_1$ then w.l.o.g. let $i_{s_0} = i_{t_0}$ and $j_{s_0} = j_{t_0}$, and choose $s = (i_s, j_{s_0}, k_{s_0})$ and $t = (i_{s_0}, j_t, k_{t_0})$ with $i_s \neq i_{s_0}$, $j_t \neq j_{s_0}$. If $t_0 \notin C_1$ then there is an $s' \in C_1 \backslash C_2$ such that $s' \cap t_0 = \emptyset$. W.l.o.g. let $s' = (i_{s_0}, j_{s_0}, k)$ where $k \neq k_{t_0}$, and choose $s = (i_{s_0}, j_{s_0}, k_{t_0})$ and $t = (i_{t_0}, j_{t_0}, k)$. Finally, select r such that $r \cap s = r \cap t = \emptyset$. Then x satisfies conditions 1-4.

(ii) Suppose $C_1 = C(s_0)$, $C_2 = (t_0, r_0)$, where $t_0 \notin C_1$, s_0 and t_0 are as above and $r_0 = (i_{r_0}, j_{r_0}, k_{r_0})$. If $r_0 \in C_1$ then let $s = r_0$, $t = t_0$ and select any r such that $r \cap s = r \cap t = \emptyset$. Suppose instead that $r_0 \notin C_1$. If $t_0 \cap s_0 \neq \emptyset$ then there is a $t' \in C_2$ such that $t' \cap s_0 = \emptyset$ and a corresponding r' such that $C_2 = C(t', r')$; thus w.l.o.g. we can assume $t_0 \cap s_0 = \emptyset$. Now, w.l.o.g., assume $i_{r_0} \neq i_{s_0}$, $j_{r_0} \neq j_{s_0}$. Then there is an $s' \in C_1$ such that $s' = (i_{s_0}, j_{s_0}, k)$ with $k \neq k_{r_0}$, $k \neq k_{t_0}$ (in particular, if $k_{r_0} \neq k_{s_0}$ then we can take $k = k_{s_0}$, otherwise there is at least one element of K which is not equal to k_{t_0} or to $k_{s_0} = k_{r_0}$, and we can choose this element to be k). Choose $s = (i_{s_0}, j_{s_0}, k_{r_0})$, $t = t_0$ and $r = (i_{r_0}, j_{r_0}, k)$ Then x satisfies conditions 1-4.

(iii) Suppose $C_1 = C(s_0, w_0)$, $C_2 = (t_0, r_0)$ where s_0, t_0 and r_0 are as above, $s_0 \notin C_2$, $t_0 \notin C_1$, $s_0 \cap t_0 = \emptyset$ and $w_0 = (i_{w_0}, j_{w_0}, k_{w_0})$. Suppose first that $w_0 \cap t_0 = \emptyset$. If $w_0 \neq r_0$, choose $s = s_0$, $t = t_0$, $r = w_0$. If $w_0 = r_0$ then $C_1 \cap C_2 = \emptyset$ so we can write $C_1 = C(s', w')$ with $s' \neq s_0$, $w' \neq r_0$. Then we choose $s = s'$, $t = t_0$, $r = w'$. Now suppose $w_0 \cap t_0 \neq \emptyset$. It follows that

$w_0 \neq r_0$. Define a new triplet w' by selecting the element or elements of $w_0 \backslash t_0$ and taking the remaining element or elements from r_0. By defintion $w' \cap t_0 = \emptyset$.

If $w' \cap s_0 = \emptyset$ and $w' \neq r_0$, choose $s = s_0$, $t = t_0$, $r = w'$.

If $w' \cap s_0 = \emptyset$ and $w' = r_0$ then choose $s = t_0$, $t = s_0$, $r = w'$, exchanging the roles of C_1 and C_2.

If $w' \cap s_0 \neq \emptyset$ then it follows from the definition of w' that $r_0 \cap s_0 \neq \emptyset$ and that the common elements are in positions in which w_0 and t_0 share common elements. W.l.o.g. suppose $i_{w'} = i_{r_0} = i_{s_0}$ and $i_{w_0} = i_{t_0}$. Then there is an $i \in I$ such that $i \neq i_{r_0} = i_{s_0}$ and $i \neq i_{w_0} = i_{t_0}$. The same holds for other elements of $w' \cap s_0$, if any. Define the triple w'' from w' by replacing $i_{w'} = i_{r_0}$ with i and doing the same for any other elements of $w' \cap s_0$. Choose $s = s_0$, $t = t_0$, $r = w''$.

For each of the above cases, the resulting x satisfies conditions 1-4. ‖

<u>Proof of Theorem 3.8.</u> Let C_1 and C_2 be distinct cliques. Then $C_1 \backslash C_2 \neq \emptyset$ and $C_2 \backslash C_1 \neq \emptyset$. From Lemma 3.9, there exists a feasible solution x such that

$$\sum_{s \in C_1} x_s = 1 \quad \text{and} \quad \sum_{t \in C_2} x_t = 1$$

We will show next how to modify that solution to produce a solution x' with

$$\sum_{s \in C_1} x'_s = 1 \quad \text{and} \quad \sum_{t \in C_2} x'_t = 0$$

We construct x' by exchanging one element of t and the corresponding element of r, denoting the triplets thus created t', r'. It follows from condition 4 of the lemma that for at least one of the three possible t', r' pairs, $t' \notin C_2$ and $r' \notin C_2$. Since $x'_j = x_j$ for all j except $x'_s = x'_{t'} = x'_{r'} = 1$ and $x'_t = x'_r = 0$, x' is feasible and satisfies the inequality corresponding to C_1 with equality and the inequality corresponding to C_2 strictly. Since x' lies on the facet corresponding to C_1 but not on the facet corresponding to C_2, it follows that C_1 and C_2 define distinct facets of P_I.||

It is easy to see that each inequality $x_s \geq 0$ also defines a facet distinct from those defined by other trivial inequalities or by clique-inequalities.

4. Detecting Violated Clique-Facets

It is of great interest in terms of algorithm development to be able to determine, for an arbitrary noninteger solution to the LP-relaxation of an integer program, whether that solution violates a facet of the convex hull of integer solutions. One may solve the LP-relaxation, then identify a facet-defining inequality that cuts off the solution obtained and either add it to the constraint set of the LP, or take it into the objective function with a Lagrange multiplier. In general, for an NP-hard problem the facet-identification problem is also NP-hard, but for some subsets of the facets it may be possible to efficiently identify which, if any, members of the subsets are violated by an LP solution. Recent efforts to implement algorithms based on this strategy (and employing branch-and-bound techniques when a fractional solution is reached that does not violate any of the facets under consideration) have met with marked success [1], [8]. In this section we describe an efficient algorithm for detecting clique-facets violated by an

arbitrary $x \varepsilon P$, i.e., an arbitrary solution to the LP-relaxation of AP3. Although the cardinality of the set of clique-facets is $O(n^6)$, (namely, n^3 facets from cliques of class 2, and $n^3(n-1)^3/4$ from cliques of class 3), the proposed algorithm can be shown to have a worst-case running time of $O(n^4)$. In terms of the number $|S|$ of variables, this is $O(|S|^{4/3})$.

We first remark that given a noninteger $x \varepsilon P$, it can be detected in $O(n^4)$ steps whether any inequality induced by a clique of class 2 is violated. Indeed, each of the n^3 cliques of class 2 is associated with some $s \varepsilon S$, and is induced by a node set of the form $\{s\} \cup T(s)$, where $T(s)$ is the set of those triplets that differ from the the triplet s in exactly one element. Since the cardinality of $T(s)$ is $3(n-1)$, for each $s \varepsilon S$ it requires $O(n)$ steps to identify and add all x_{ijk} such that $(i,j,k) \varepsilon C(s)$, in order to check whether the sum exceeds 1 (in which case the corresponding inequality is violated) or not. To execute this for all $s \varepsilon S$ therefore requires $O(n \times n^3) = O(n^4)$ steps.

For cliques of class 3 (whose number is $O(n^6)$) the complexity bound is not so straightforward. However, we will give an algorithm which performs that task too in $O(n^4)$ steps. This is possible due to the following fact: Each clique of class 3 is of cardinality 4; therefore any $x \varepsilon P$ that violates some inequality induced by a clique of class 3 must have at least one component of value $\geq 1/4$. On the other hand, we have

<u>Lemma 4.1</u> For any $x \varepsilon P$ and any positive integer k, the number of components with value $\geq 1/k$ is $\leq kn$.

<u>Proof</u>. The value of the linear program

(L) max $\{ex \mid x \varepsilon P\}$

is easily seen to be n, since the vectors $x \varepsilon R^{n^3}$ and $u \varepsilon R^{3n}$, defined by

$$x_s = 1/n^2, \forall s \varepsilon S$$

and

$u_r = 1/3$, $\forall$ r ϵ R,

are feasible solutions to (L) and its dual, respectively, with the common value of n; hence they are optimal.

Now if x has more than kn components with value $\geq 1/k$, then ex > n, a contradiction.||

<u>Theorem 4.2</u>. It can be determined in $O(n^4)$ steps whether a given x ϵ P violates a facet defining inequality induced by a clique of class 3.

<u>Proof</u>. Let C(s,t) be the node set of a clique of class 3. Since $|C(s,t)| = 4$, if x ϵ P violates the facet-inequality corresponding to C(s,t), then from Lemma 4.1 x has at least one component $\geq 1/4$. Further, if $C(s,t) = \{s,t_1,t_2,t_3\}$, from Proposition 2.4 there is no loss of generality in assuming that this happens for the component indexed by s, i.e., that $x_s \geq 1/4$. Thus, instead of examining all ordered pairs (s,t) such that $a^s \cdot a^t = 0$, we can restrict ourselves to examine those ordered pairs (s,t) such that $x_s \geq 1/4$ and $a^s \cdot a^t = 0$.

Consider now the following algorithm.

1. Order S according to nonincreasing values of x_s, sϵS.

2. For each of the first 4n elements $s = (i_s, j_s, k_s)$ of the ordered set S such that $x_s > 1/4$ and each of the $(n-1)^3$ triplets $t = (i_t, j_t, k_t)$ ϵ S such that $i_t \neq i_s$, $j_t \neq j_s$ and $k_t \neq k_s$, calculate the sum $\Sigma(s,t) = x_{i_s j_s k_s} + x_{i_s j_t k_t} + x_{i_t j_s k_t} + x_{i_t j_t k_s}$. If $\Sigma(s,t) > 1$, stop: the inequality associated with (s,t) is violated; otherwise continue.

Since the algorithm examines all pairs (s,t) such that $a^s \cdot a^t = 0$ and $x_s > 1/4$, it either finds a pair whose corresponding facet inequality is violated by x, or it stops with the conclusion that x satisfies all facet-inequalities induced by cliques of class 3. Step 1 is executed once and it

requires $O(n^3 \log n^3)$ operations. Step 2 is executed at most $4n(n-1)^3$ times, and each execution requires 3 additions. Hence, the overall complexity of the algorithm is $O(n^4)$.||

References

[1] Balas, E. and Christofides, N., "A Restricted Lagrangian Approach to the Traveling Salesman Problem." Mathematical Programming, 21, 1981, 19-46.

[2] Balas, E. and Landweer, P., "Traffic Assignment in Communication Satellites." Operations Research Letters, 2, 1983, 141-147.

[3] Balas, E. and Padberg, M., "Set Partitioning: A Survey." SIAM Review, 18, 1976, 710-760.

[4] Corban, A., "A Multidimensional Transportation Problem." Revue Roumaine des Mathematiques Pures et Appliquees, 9, 1964, 721-735.

[5] Corban, A., "On a Three-Dimensional Transportation Problem." Revue Roumaine des Mathematiques Pures et Appliquees, 11, 1966, 57-75.

[6] Cornuejols, G. and Pulleyblank, W., "The Traveling Salesman Polytope and {0,2}-Matchings." Annals of Discrete Mathematics, 16, 1982, 27-55.

[7] Cornuejols, G. and Thizy, J-M., "Some Facets of the Simple Plant Location Polytope." Mathematical Programming, 23, 1982, 50-74.

[8] Crowder, H. and Padberg, M., "Solving Large-Scale Symmetric Traveling Salesman Problems to Optimality." Management Science, 26, 1980, 495-509.

[9] Frieze, A. M., "Complexity of a 3-Dimensional Assignment Problem." European Journal of Operations Research, 13, 1983, 161-164.

[10] Frieze, A. M. and Yadegar, J., "An Algorithm for Solving 3-Dimensional Assignment Problems with Application to Scheduling a Teaching Practice." Journal of the Operational Research Society, 32, 1981, 989-995.

[11] Garey, M. R. and Johnson, D. S., Computers and Intractibility: A Guide to the Theory of NP-Completeness. W. H. Freeman and Company, San Francisco, 1979.

[12] Gotlieb, C. C., "The Construction of Class-Teacher Timetables." Proceedings of the IFIP Congress, 1962, 73-77.

[13] Haley, K. B., "The Solid Transportation Problem." Operations Research, 10, 1962, 448-463.

[14] Haley, K. B., "The Multi-Index Problem." Operations Research, 11, 1963, 368-379.

[15] Leue, O., "Methoden zur Losung Dreidimensionaler Zuodnungsprobleme." Angewandte Informatik, April 1972, 154-162.

[16] Maurras, J. F., "Polytopes a sommets dans $\{0,1\}^n$." Ph.D. Dissertation, University of Paris, 1976.

[17] Padberg, M., "On the Facial Structure of Set Packing Polyhedra." Mathematical Programming, 5, 1973, 199-215.

[18] Pierskalla, W. P., "The Tri-Substitution Method for the Three-Dimensional Assignment Problem." CORS Journal, 5, 1967, 71-81.

[19] Pierskalla, W. P., The Multidimensional Assignment Problem." Operations Research, 16, 1968, 422-431.

[20] Schell, E. D., "Distribution of a Product by Several Properties." Second Symposium in Linear Programming, 2, DCS/Comptroller HQ. U.S. Air Force, Washington, D.C., 1955, 615-642.

[21] Vlach, M., "Branch and Bound Method for the Three-Index Assignment Problem." Ekonomicko-Matematicky Obzor (Czechoslovakia), 3, 1967, 181-191.

4. TITLE (and Subtitle)	5. TYPE OF REPORT & PERIOD COVERED
SOME FACETS OF THE TRI-INDEX ASSIGNMENT POLYTOPE	October 1984
	6. PERFORMING ORG. REPORT NUMBER
	MSRR 511

7. AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
Egon Balas	N00014-82-K-0329
Matthew J. Saltzman	NR 047-048

9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Graduate School of Industrial Administration	
Carnegie-Mellon University	
Pittsburgh, Pennsylvania 15213	

11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE
Personnel and Training Research Programs	April 1984
Office of Naval Research (Code 434)	13. NUMBER OF PAGES
Arlington, VA 22217	34

14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)	15. SECURITY CLASS. (of this report)
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, If different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

3-Index assignment problem	Facets
3-Dimensional Matching problem	Cliques
Polyhedral Combinatorics	Intersection Graph

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Given three disjoint n-sets and the family of all weighted triplets that contain exactly one element of each set, the 3- index assignment (or 3- dimensional matching) problem asks for a minimum-weight subcollection of triplets that covers exactly (i.e., partitions) the union of the three sets. Unlike the common (2-index) assignment problem, the 3-index problem is NP-complete. In this paper we examine the facial structure of the 3-index assignment polytope (the convex hull of feasible solutions to the problem) with the aid of

ABSTRACT (continued)

the intersection graph of the coefficient matrix of the problem's constraint set.
In particular, we describe the cliques of the intersection graph as belonging to
three distinct classes, and show that cliques in three of the three classes induce
inequalities that define facets of our polytope. Furthermore, we given an $O(n^3)$
procedure (note that the number of variables is n^3) for finding a facet-defining
clique-inequality violated by a given noninteger solution to the linear programming
relaxation of the 3-index assignment problem, or showing that no such inequality
exists.

END

FILMED

2-85

DTIC